

# QUANTUM DECOHERENCE OF THE DAMPED HARMONIC OSCILLATOR

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## Abstract

In the framework of the Lindblad theory for open quantum systems, we determine the degree of quantum decoherence of a harmonic oscillator interacting with a thermal bath. It is found that the system manifests a quantum decoherence which is more and more significant in time. We also calculate the decoherence time and show that it has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations.

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## 1 Introduction

The quantum to classical transition and classicality of quantum systems continue to be among the most interesting problems in many fields of physics, for both conceptual and experimental reasons [1, 2]. Two conditions are essential for the classicality of a quantum system [3]: a) quantum decoherence (QD), that means the irreversible, uncontrollable and persistent formation of a quantum correlation (entanglement) of the system with its environment [4], expressed by the damping of the coherences present in the quantum state of the system, when the off-diagonal elements of the density matrix decay below a certain level, so that this density matrix becomes approximately diagonal and b) classical correlations, expressed by the fact that the Wigner function of the quantum system has a peak which follows the classical equations of motion in phase space with a good degree of approximation, that is the quantum state becomes peaked along a classical trajectory. Classicality is an emergent property of open quantum systems, since both main features of this process – QD and classical correlations – strongly depend on the interaction between the system and its external environment [1, 2].

The role of QD became relevant in many interesting physical problems. In many cases one is interested in understanding QD because one wants to prevent decoherence from damaging quantum states and to protect the information stored in quantum states from the degrading effect of the interaction with the environment. Decoherence is also responsible for washing out the quantum interference effects which are desirable

to be seen as signals in experiments. QD has a negative influence on many areas relying upon quantum coherence effects, in particular it is a major problem in quantum optics and physics of quantum information and computation [5].

In this work we study QD of a harmonic oscillator interacting with an environment, in particular with a thermal bath, in the framework of the Lindblad theory for open quantum systems. We determine the degree of QD and then we apply the criterion of QD. We consider different regimes of the temperature of environment and it is found that the system manifests a QD which in general increases with time and temperature.

The organizing of the paper is as follows. In Sec. 2 we review the Lindblad master equation for the damped harmonic oscillator and solve the master equation in coordinate representation. Then in Sec. 3 we investigate QD and in Sec. 4 we calculate the decoherence time of the system. We show that this time has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations. A summary and concluding remarks are given in Sec. 5.

## 2 Master equation and density matrix

In the Lindblad axiomatic formalism based on quantum dynamical semigroups, the irreversible time evolution of an open system is described by the following general quantum Markovian master equation for the density operator  $\rho(t)$  [6]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (1)$$

The harmonic oscillator Hamiltonian  $H$  is chosen of the general quadratic form

$$H = H_0 + \frac{\mu}{2}(qp + pq), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \quad (2)$$

and the operators  $V_j, V_j^\dagger$ , which model the environment, are taken as linear polynomials in coordinate  $q$  and momentum  $p$ . Then the master equation (1) takes the following form [7]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[q, \rho p + p\rho] + \frac{i}{2\hbar}(\lambda - \mu)[p, \rho q + q\rho] \\ & - \frac{D_{pp}}{\hbar^2}[q, [q, \rho]] - \frac{D_{qq}}{\hbar^2}[p, [p, \rho]] + \frac{D_{pq}}{\hbar^2}([q, [p, \rho]] + [p, [q, \rho]]). \end{aligned} \quad (3)$$

The diffusion coefficients  $D_{pp}, D_{qq}, D_{pq}$  and the dissipation constant  $\lambda$  satisfy the fundamental constraints:  $D_{pp} > 0, D_{qq} > 0$  and  $D_{pp}D_{qq} - D_{pq}^2 \geq \lambda^2\hbar^2/4$ . In the particular

case when the asymptotic state is a Gibbs state  $\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tr} e^{-\frac{H_0}{kT}}$ , these coefficients become

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (4)$$

where  $T$  is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if  $\lambda > \mu$  and

$$(\lambda^2 - \mu^2) \coth^2 \frac{\hbar \omega}{2kT} \geq \lambda^2 \quad (5)$$

and the asymptotic values  $\sigma_{qq}(\infty)$ ,  $\sigma_{pp}(\infty)$ ,  $\sigma_{pq}(\infty)$  of the dispersion (variance), respectively correlation (covariance), of the coordinate and momentum, reduce to [7]

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar \omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m \omega}{2} \coth \frac{\hbar \omega}{2kT}, \quad \sigma_{pq}(\infty) = 0. \quad (6)$$

We consider a harmonic oscillator with an initial Gaussian wave function ( $\sigma_q(0)$  and  $\sigma_p(0)$  are the initial averaged position and momentum of the wave packet)

$$\Psi(q) = \left( \frac{1}{2\pi\sigma_{qq}(0)} \right)^{\frac{1}{4}} \exp \left[ -\frac{1}{4\sigma_{qq}(0)} \left( 1 - \frac{2i}{\hbar} \sigma_{pq}(0) \right) (q - \sigma_q(0))^2 + \frac{i}{\hbar} \sigma_p(0) q \right], \quad (7)$$

representing a correlated coherent state (squeezed coherent state) with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0) = \frac{\hbar \delta}{2m\omega}, \quad \sigma_{pp}(0) = \frac{\hbar m \omega}{2\delta(1-r^2)}, \quad \sigma_{pq}(0) = \frac{\hbar r}{2\sqrt{1-r^2}}. \quad (8)$$

Here  $\delta$  is the squeezing parameter which measures the spread in the initial Gaussian packet and  $r$ , with  $|r| < 1$  is the correlation coefficient. The initial values (8) correspond to a minimum uncertainty state, since they fulfil the generalized uncertainty relation  $\sigma_{qq}(0)\sigma_{pp}(0) - \sigma_{pq}^2(0) = \hbar^2/4$ . For  $\delta = 1$  and  $r = 0$  the correlated coherent state becomes a Glauber coherent state.

From Eq. (3) we derive the evolution equation in coordinate representation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2} \right) \rho - \frac{im\omega^2}{2\hbar} (q^2 - q'^2) \rho \\ &- \frac{1}{2} (\lambda + \mu) (q - q') \left( \frac{\partial}{\partial q} - \frac{\partial}{\partial q'} \right) \rho + \frac{1}{2} (\lambda - \mu) [(q + q') \left( \frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right) + 2] \rho \\ &- \frac{D_{pp}}{\hbar^2} (q - q')^2 \rho + D_{qq} \left( \frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right)^2 \rho - 2iD_{pq}\hbar(q - q') \left( \frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right) \rho. \end{aligned} \quad (9)$$

The first two terms on the right-hand side of this equation generate the usual Liouvillean unitary evolution. The third and forth terms are the dissipative terms and have

a damping effect (exchange of energy with environment). The last three are noise (diffusive) terms and produce fluctuation effects in the evolution of the system.  $D_{pp}$  promotes diffusion in momentum and generates decoherence in coordinate  $q$  – it reduces the off-diagonal terms, responsible for correlations between spatially separated pieces of the wave packet. Similarly  $D_{qq}$  promotes diffusion in coordinate and generates decoherence in momentum  $p$ . The  $D_{pq}$  term is the so-called "anomalous diffusion" term and it does not generate decoherence.

The density matrix solution of Eq. (9) has the general Gaussian form

$$\begin{aligned} < q | \rho(t) | q' > = & \left( \frac{1}{2\pi\sigma_{qq}(t)} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2\sigma_{qq}(t)} \left( \frac{q+q'}{2} - \sigma_q(t) \right)^2 \right. \\ & \left. - \frac{\sigma(t)}{2\hbar^2\sigma_{qq}(t)} (q-q')^2 + \frac{i\sigma_{pq}(t)}{\hbar\sigma_{qq}(t)} \left( \frac{q+q'}{2} - \sigma_q(t) \right) (q-q') + \frac{i}{\hbar} \sigma_p(t) (q-q') \right], \end{aligned} \quad (10)$$

where  $\sigma(t) \equiv \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t)$  is the Schrödinger generalized uncertainty function. In the case of a thermal bath we obtain the following steady state solution for  $t \rightarrow \infty$  ( $\epsilon \equiv \hbar\omega/2kT$ ):

$$< q | \rho(\infty) | q' > = \left( \frac{m\omega}{\pi\hbar\coth\epsilon} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m\omega}{4\hbar} \left[ \frac{(q+q')^2}{\coth\epsilon} + (q-q')^2 \coth\epsilon \right] \right\}. \quad (11)$$

### 3 Quantum decoherence

An isolated system has an unitary evolution and the coherence of the state is not lost – pure states evolve in time only to pure states. The QD phenomenon, that is the loss of coherence or the destruction of off-diagonal elements representing coherences between quantum states in the density matrix, can be achieved by introducing an interaction between the system and environment: an initial pure state with a density matrix which contains nonzero off-diagonal terms can non-unitarily evolve into a final mixed state with a diagonal density matrix.

Using new variables  $\Sigma = (q+q')/2$  and  $\Delta = q-q'$ , the density matrix (10) becomes

$$\rho(\Sigma, \Delta, t) = \sqrt{\frac{\alpha}{\pi}} \exp \left[ -\alpha\Sigma^2 - \gamma\Delta^2 + i\beta\Sigma\Delta + 2\alpha\sigma_q(t)\Sigma + i \left( \frac{\sigma_p(t)}{\hbar} - \beta\sigma_q(t) \right) \Delta - \alpha\sigma_q^2(t) \right], \quad (12)$$

with the abbreviations

$$\alpha = \frac{1}{2\sigma_{qq}(t)}, \quad \gamma = \frac{\sigma(t)}{2\hbar^2\sigma_{qq}(t)}, \quad \beta = \frac{\sigma_{pq}(t)}{\hbar\sigma_{qq}(t)}. \quad (13)$$

The representation-independent measure of the degree of QD [3] is given by the ratio of the dispersion  $1/\sqrt{2\gamma}$  of the off-diagonal element  $\rho(0, \Delta, t)$  to the dispersion  $\sqrt{2/\alpha}$  of the diagonal element  $\rho(\Sigma, 0, t)$  :

$$\delta_{QD}(t) = \frac{1}{2} \sqrt{\frac{\alpha}{\gamma}} = \frac{\hbar}{2\sqrt{\sigma(t)}}. \quad (14)$$

The finite temperature Schrödinger generalized uncertainty function has the expression [8] (with the notation  $\Omega^2 \equiv \omega^2 - \mu^2$ ,  $\omega > \mu$ )

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[ 1 - \left( \delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right] \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[ \left( \delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} \right. \\ & \left. \left. + \left( \delta - \frac{1}{\delta(1-r^2)} \right) \frac{\mu \sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1-\cos(2\Omega t))}{\Omega^2\sqrt{1-r^2}} \right] + \coth^2 \epsilon \right\}. \end{aligned} \quad (15)$$

In the limit of long times Eq. (15) yields  $\sigma(\infty) = (\hbar^2 \coth^2 \epsilon)/4$ , so that we obtain

$$\delta_{QD}(\infty) = \tanh \frac{\hbar\omega}{2kT}, \quad (16)$$

which for high  $T$  becomes  $\delta_{QD}(\infty) = \hbar\omega/2kT$ . We see that  $\delta_{QD}$  decreases, and therefore QD increases, with time and temperature, i.e. the density matrix becomes more and more diagonal at higher  $T$  and the contributions of the off-diagonal elements get smaller and smaller. At the same time the degree of purity decreases and the degree of mixedness increases with  $T$ . For  $T = 0$  the asymptotic (final) state is pure and  $\delta_{QD}$  reaches its initial maximum value 1.  $\delta_{QD} = 0$  when the quantum coherence is completely lost, and if  $\delta_{QD} = 1$  there is no QD. Only if  $\delta_{QD} < 1$  we can say that the considered system interacting with the thermal bath manifests QD, when the magnitude of the elements of the density matrix in the position basis are peaked preferentially along the diagonal  $q = q'$ . Dissipation promotes quantum coherences, whereas fluctuation (diffusion) reduces coherences and promotes QD. The balance of dissipation and fluctuation determines the final equilibrium value of  $\delta_{QD}$ . The initial pure state evolves approximately following the classical trajectory in phase space and becomes a quantum mixed state during the irreversible process of QD.

## 4 Decoherence time

In order to obtain the expression of the decoherence time, we consider the coefficient  $\gamma$  (13), which measures the contribution of non-diagonal elements in the density matrix

(12). For short times ( $\lambda t \ll 1, \Omega t \ll 1$ ), we have:

$$\gamma(t) = -\frac{m\omega}{4\hbar\delta} \left\{ 1 + 2[\lambda(\delta + \frac{r^2}{\delta(1-r^2)}) \coth \epsilon + \mu(\delta - \frac{r^2}{\delta(1-r^2)}) \coth \epsilon - \lambda - \mu - \frac{\omega r}{\delta\sqrt{1-r^2}}]t \right\}. \quad (17)$$

The quantum coherences in the density matrix decay exponentially and the decoherence time scale is given by

$$t_{deco} = \frac{1}{2[\lambda(\delta + \frac{r^2}{\delta(1-r^2)}) \coth \epsilon + \mu(\delta - \frac{r^2}{\delta(1-r^2)}) \coth \epsilon - \lambda - \mu - \frac{\omega r}{\delta\sqrt{1-r^2}}]}. \quad (18)$$

The decoherence time depends on the temperature  $T$  and the coupling  $\lambda$  (dissipation coefficient) between the system and environment, the squeezing parameter  $\delta$  and the initial correlation coefficient  $r$ . We notice that the decoherence time is decreasing with increasing dissipation, temperature and squeezing.

For  $r = 0$  we obtain:

$$t_{deco} = \frac{1}{2(\lambda + \mu)(\delta \coth \epsilon - 1)} \quad (19)$$

and at temperature  $T = 0$  (then we have to take  $\mu = 0$ ), this becomes

$$t_{deco} = \frac{1}{2\lambda(\delta - 1)}. \quad (20)$$

We see that when the initial state is the usual coherent state ( $\delta = 1$ ), then the decoherence time tends to infinity. This corresponds to the fact that for  $T = 0$  and  $\delta = 1$  the coefficient  $\gamma$  is constant in time, so that the decoherence process does not occur in this case.

At high temperature, expression (18) becomes ( $\tau \equiv 2kT/\hbar\omega$ )

$$t_{deco} = \frac{1}{2[\lambda(\delta + \frac{r^2}{\delta(1-r^2)}) + \mu(\delta - \frac{r^2}{\delta(1-r^2)})]\tau}. \quad (21)$$

If, in addition  $r = 0$ , then we obtain

$$t_{deco} = \frac{\hbar\omega}{4(\lambda + \mu)\delta kT}. \quad (22)$$

The generalized uncertainty function  $\sigma(t)$  (15) has the following behaviour for short times:

$$\sigma(t) = \frac{\hbar^2}{4} \left\{ 1 + 2[\lambda(\delta + \frac{1}{\delta(1-r^2)}) \coth \epsilon + \mu(\delta - \frac{1}{\delta(1-r^2)}) \coth \epsilon - 2\lambda]t \right\}. \quad (23)$$

This expression shows explicitly the contribution for small time of uncertainty that is intrinsic to quantum mechanics, expressed through the Heisenberg uncertainty principle and uncertainty due to the coupling to the thermal environment. From Eq. (23) we can determine the time  $t_d$  when thermal fluctuations become comparable with quantum fluctuations. At high temperature we obtain

$$t_d = \frac{1}{2\tau[\lambda(\delta + \frac{1}{\delta(1-r^2)}) + \mu(\delta - \frac{1}{\delta(1-r^2)})]}. \quad (24)$$

By thermal fluctuations we mean the fluctuations that arise in the generalized uncertainty function  $\sigma(t)$  from the coupling of the harmonic oscillator to the thermal bath at arbitrary temperature  $T$ , even at  $T = 0$ . By quantum fluctuations we mean fluctuations of the quantum harmonic oscillator at zero coupling with the thermal bath.

As expected, we see that the decoherence time  $t_{deco}$  has the same scale as the time  $t_d$  after which thermal fluctuations become comparable with quantum fluctuations [9, 10]. The values of  $t_{deco}$  and  $t_d$  become closer with increasing temperature and squeezing.

When  $t \gg t_{rel}$ , where  $t_{rel} \approx \lambda^{-1}$  is the relaxation time, which governs the rate of energy dissipation, the particle reaches equilibrium with the environment. In the macroscopic domain QD occurs very much faster than relaxation, so that for all macroscopic bodies the dissipation term becomes important much later after the decoherence term has already dominated and diminished the off-diagonal terms of the density matrix. We remark also that  $t_{deco}$  can be of the order of  $t_{rel}$  for sufficiently low temperatures and small wave packet spread (small squeezing coefficient).

## 5 Summary and concluding remarks

We have studied QD with the Markovian equation of Lindblad for a system consisting of an one-dimensional harmonic oscillator in interaction with a thermal bath in the framework of the theory of open quantum systems based on quantum dynamical semigroups.

(1) Using the criterion of QD for the considered model, we have shown that QD in general increases with time and temperature. For large temperatures, QD is strong and the degree of mixedness is high, while for zero temperature the asymptotic final state is pure. With increasing squeezing parameter and initial correlation, QD becomes stronger, but the asymptotic value of the degree of QD does not depend on the initial

squeezing and correlation, it depends on temperature only. QD is expressed by the loss of quantum coherences in the case of a thermal bath at finite temperature.

(2) We determined the general expression of the decoherence time, which shows that it is decreasing with increasing dissipation, temperature and squeezing. We have also shown that the decoherence time has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations and the values of these scales become closer with increasing temperature and squeezing. After the decoherence time, the decohered system is not necessarily in a classical regime. There exists a quantum statistical regime in between. Only at a sufficiently high temperature the system can be considered in a classical regime.

The Lindblad theory provides a self-consistent treatment of damping as a general extension of quantum mechanics to open systems and gives the possibility to extend the model of quantum Brownian motion. The results obtained in the framework of this theory are a useful basis for the description of the connection between uncertainty, decoherence and correlations (entanglement) of open quantum systems with their environment, in particular in the study of Gaussian states of continuous variable systems used in quantum information processing to quantify the similarity or distinguishability of quantum states using distance measures, like trace distance and quantum fidelity.

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